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Questions and Answers

This department will provide a forum in which individuals can advertise problems, sometimes of a philosophical and foundational nature, which are bothering them and for which they have not been able to discover a satisfactory answer. Paradoxes, methods of pedagogical exposition, and many other subjects can be presented in the form of brief communications and letters. Such notes need not represent even partial research papers.

Contributions should be sent to the Editor of the Department:

Dr. Frank H. Stillinger, Jr.
Chemical Physics Research Department
Bell Telephone Laboratories, Inc.
Murray Hill, New Jersey 07971

Let $n(t)$ be a white Gaussian noise (derivative of a Wiener process) process, with unit power. Define

$$x(t) = \exp\left[\int_0^t n(\zeta)\, d\zeta\right]$$

What is the derivative of the $x(t)$ process? By the established rules of calculus, the answer would be

$$\frac{d}{dt} x(t) = x(t)\, n(t)$$

But, taking difference quotients,

$$\frac{x(t+\Delta) - x(t)}{\Delta} = x(t)\, \frac{\exp[\int_t^{t+\Delta} n(\zeta)\, d\zeta] - 1}{\Delta}$$

$$= x(t)\left[\frac{1}{\Delta}\int_t^{t+\Delta} n(\zeta)\, d\zeta + \frac{1}{2}\frac{1}{\Delta}\left(\int_t^{t+\Delta} n(\zeta)\, d\zeta\right)^2 + \text{higher-order terms}\right]$$

By the properties of white noise, where n is the derivative of a Wiener process, it follows that as Δ goes to zero

$$\frac{1}{\Delta}\left[\int_{t}^{t+\Delta} n(\zeta)\, d\zeta\right]^{2} \to 1$$

This would mean

$$\frac{dx}{dt} = n(t)\, x(t) + \tfrac{1}{2}$$

Which is correct? In particular, which one should we use in characterizing $x(t)$ by means of a differential equation?

A. V. Balakrischnan

College of Engineering
University of California, Los Angeles